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FAST TRACK COMMUNICATION

Genus two finite gap solutions to the vector nonlinear Schrödinger equation

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Abstract

A recently published article presents a technique used to derive explicit formulae for odd genus solutions to the vector nonlinear Schrödinger equation. In another article solutions of genus two are derived using a different approach which assumes a separable ansatz. In this communication, the extension of the first technique to the even genus case is discussed, and this extension is carried out explicitly for genus two. Furthermore, a birational mapping is found between the spectral curves that arise in the two approaches.

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In a recent paper [1], Elgin *et al* provide an effective algorithm for determining finite gap solutions to the vector nonlinear Schrödinger equation (VNLS) in an anomalously dispersive regime,

$$\mathrm{i}\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\boldsymbol{q}\boldsymbol{q}^{\dagger}\boldsymbol{q} = 0. \tag{1}$$

The VNLS is an envelope equation which models the propagation of ultra-short light pulses and continuous-wave beams along optical fibres. Here $\boldsymbol{q}(x,t) = (q_1, q_2)^{\mathsf{T}}$ is a complex-valued vector representing the electromagnetic field components of the envelope and $\boldsymbol{q}^{\dagger}$ is its Hermitian conjugate.

In [1], finite gap solutions to (1) are explicitly constructed using an algebrogeometric technique first devised by Krichever [2]. A hierarchy of mutually commuting flows is constructed, containing the VNLS flow. Stationary solutions of other flows in the hierarchy yield finite gap solutions of the VNLS itself, expressed in terms of theta functions on the desingularization of the spectral curve. In constructing the hierarchy, Elgin *et al* set certain constants of integration equal to zero, and in doing so restrict the method to solutions of odd genus.

In [3], Christiansen *et al* apply a separable ansatz to find solutions to the same equation. These solutions are expressed in terms of generalized Weierstrass $\wp$ functions on a spectral curve of genus two. Wright [4] also demonstrates the existence of stationary solutions to (1) of arbitrary genus, suggesting that it should be possible to apply the method of [1] to the even genus case.

In this communication, we show how the method of [1] can be modified to yield solutions expressed in terms of theta functions on a Riemann surface of genus two. We then show that these solutions are indeed the same as those found in [3], and that the spectral curves that arise in the two approaches are birationally equivalent.

Consider the Manakov Lax pair system:

$$\begin{aligned} \boldsymbol{v}_x &= \mathcal{L}_1(z)\boldsymbol{v} = (zL_0 + L_1)\boldsymbol{v} \\ \boldsymbol{v}_t &= \mathcal{L}_2(z)\boldsymbol{v} = (z^2L_0 + zL_1 + L_2)\boldsymbol{v} \end{aligned} \tag{2}$$

where

$$L_0 = \begin{pmatrix} -\frac{\mathrm{i}}{2} & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & \frac{\mathrm{i}}{2}\mathbb{I}_2 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & \boldsymbol{q}^\mathsf{T} \\ -\overline{\boldsymbol{q}} & \mathbb{O}_2 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} \mathrm{i}\boldsymbol{q}^\dagger\boldsymbol{q} & \mathrm{i}\boldsymbol{q}_x^\mathsf{T} \\ \mathrm{i}\overline{\boldsymbol{q}}_x & -\mathrm{i}\overline{\boldsymbol{q}}\boldsymbol{q}^\mathsf{T} \end{pmatrix}$$

are 3×3 matrices. Here, $\boldsymbol{0}$ is the zero 2-vector, $\mathbb{I}_2$ is the 2×2 identity matrix and $\mathbb{O}_2$ is the 2×2 zero matrix whilst † represents the Hermitian conjugate. Applying the consistency condition $\boldsymbol{v}_{xt} = \boldsymbol{v}_{tx}$ and inspecting coefficients of order z^0 recovers equation (1). As Elgin *et al* point out, the spectral curve

$$\{(z, \mu) \in \mathbb{C}^2 : \det(\mathcal{L}_2(z) - \mu\mathbb{I}_3) = 0\} \tag{3}$$

of stationary solutions to this system has genus one. By introducing constants appended to the terms of L_2, and considering the stationary solutions of this modified Lax system, we obtain a different solution to (1). Moreover, a consequence of the modification is that the spectral curve of this solution has genus two.

Consider replacing L_2 above by

$$L_2 = \begin{pmatrix} \mathrm{i}\boldsymbol{q}^\dagger\boldsymbol{q} & \mathrm{i}\boldsymbol{q}_x^\mathsf{T} \\ \mathrm{i}\overline{\boldsymbol{q}}_x & -\mathrm{i}\overline{\boldsymbol{q}}\boldsymbol{q}^\mathsf{T} \end{pmatrix} + \mathrm{i}\begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}, \tag{4}$$

with L_0 and L_1 remaining unchanged. The constants, C_{kk}, are effectively constants of integration, are all real, and are chosen such that $C_{11} = -(C_{22} + C_{33})$, so L_2 remains traceless. It is precisely these constants that are set to zero in [1]. Since our modification changes the second of equations (2), we replace t by τ in the modified system. The consistency condition $\boldsymbol{v}_{x\tau} = \boldsymbol{v}_{\tau x}$ is then equivalent to

$$\mathrm{i}\boldsymbol{q}_\tau + \boldsymbol{q}_{xx} + 2\boldsymbol{q}\boldsymbol{q}^\dagger\boldsymbol{q} - \begin{pmatrix} a_1q_1 \\ a_2q_2 \end{pmatrix} = 0 \tag{5}$$

with $a_1 = C_{22} - C_{11}$ and $a_2 = C_{33} - C_{11}$.

The polynomial defining the curve takes on a much simpler form under the regular mapping $w = \mu - \frac{\mathrm{i}}{2}z^2$. This is given by

$$f(z, w) = w^2(w + \mathrm{i}z^2) + wP(z) + Q(z) \tag{6}$$

where $P(z) = \rho_2z^2 + \rho_1z + \rho_0$ and $Q(z) = \eta_2z^2 + \eta_1z + \eta_0$ are quadratics in z whose coefficients, although functionally dependent on $q(x, t)$ and its derivatives, are independent of x and t. Setting $C_{kk} = 0$ the coefficients ρ_2, η_2 and η_1 vanish.

In the context of algebraic geometry, it is the affine complex algebraic curve

$$C_A = \{(z, w) \in \mathbb{C}^2 : f(z, w) = 0\} \tag{7}$$

that is of interest. However, in order to characterize C_A properly, it is necessary to consider the homogenization $F(Z, W, X)$ of $f(z, w)$ and to define the projective complex algebraic curve

$$C_P = \{[Z : W : X] \in \mathbb{CP}^2 : F(Z, W, X) = 0\}. \tag{8}$$

A singular point of C_P is a point at which the gradient of $F(Z, W, X)$ vanishes, $(F_Z, F_W, F_X)^\top = \mathbf{0}$. It is easy to check that there is only one such point, at $[Z : W : X] = [0 : 1 : 0]$, which we will refer to as $\infty \in C_P$ as it is the 'extra' point added to the affine curve C_A in order to obtain C_P.

There exists a resolution of singularities for C_P, that is to say there exists a compact Riemann surface M and a surjective continuous map $\pi : M \to C_P$ such that $\pi^{-1}(\infty)$ is a finite set of points in M and, setting $\hat{M} = M \backslash \pi^{-1}(\infty)$, the restriction $\pi|_{\hat{M}} : \hat{M} \to C_P \backslash \infty$ is a holomorphic bijection [5, 6]. The genus of the singular curve C_P is defined to be that of M. We can think of M—also called the desingularization of C_P—as C_P with the singular point removed and replaced by three ordinary points. We shall label these points ∞_1, ∞_2 and ∞_3, and refer to them as the points at infinity on M.

We are now in a position to calculate the genus g of M. The map $\phi : (z, w) \mapsto z$ is a three-sheeted covering of $\mathbb{C}$ by C_A. We can extend this to a three-sheeted covering Φ of $\mathbb{CP}^1$ by M which maps the points at infinity ∞_1, ∞_2 and ∞_3 on M (each on a different sheet of Φ) to $\infty \in \mathbb{CP}^1$. The points at infinity on M are not ramification points of Φ. The number of ramification points of Φ on M is therefore equal to the number of ramification points B of ϕ on C_A. This we can easily calculate by considering the resultant of $f(z, w)$ and $\frac{\partial f(z,w)}{\partial w}$ with respect to w. The resultant is a polynomial in z whose degree is equal to B. For the case (6) one can see that $B = 8$, provided

$$\Delta = \rho_2^2 - 4\mathrm{i}\eta_2 \neq 0, \tag{9}$$

and thus by the Riemann–Hurwitz formula verify that $g = 2$. Accordingly we introduce a canonical homology basis of cycles a_1, a_2, b_1 and b_2 on M [10].

Insight may be gained into how the introduction of the constants C_{kk} increases the genus by considering the limiting case $C_{kk} \to 0$. In this limit, a pair of ramification points tend to the singular point $\infty \in C_P$ meaning the desingularization M has one handle fewer and its genus is lower by one. Note that here we are using a three-sheeted covering of M, in order that the method of [1] can be applied. In fact we shall see that the Riemann surface M is hyperelliptic, i.e. there exists a *two* sheeted covering of $\mathbb{CP}^1$ by M.

In performing calculations on M we shall use local parameter $\xi = z^{-1}$ near the points at infinity, where the behaviour of w is given by

$$w = \begin{cases} -\mathrm{i}\xi^{-2} - \mathrm{i}\rho_2 - \mathrm{i}\rho_1\xi + O(\xi^2) & \text{at } \infty_1 \\ a_0^{(2,3)} + a_1^{(2,3)}\xi + O(\xi^2) & \text{at } \infty_{2,3} \end{cases} \tag{10}$$

with

$$a_0^{(2,3)} = \mathrm{i}(\rho_2 \pm \sqrt{\Delta})/2,$$
$$a_1^{(2,3)} = \pm\big(\rho_1 a_0^{(2,3)} + \eta_1\big)\big/\sqrt{\Delta}.$$

A set of holomorphic differentials for this curve can be found using the Maple program of Deconinck and Van Hoeij [7]:

$$\mathrm{d}\nu_1 = \frac{\mathrm{i}}{f_w(z, w)}\,\mathrm{d}z, \qquad \mathrm{d}\nu_2 = \frac{w}{f_w(z, w)}\,\mathrm{d}z, \tag{11}$$

in terms of which a normalized set, satisfying $\int_{a_k} \mathrm{d}\omega_j = 2\pi \mathrm{i}\delta_{jk}$, is given by

$$\mathrm{d}\omega_j = 2\pi\mathrm{i}\sum_{k=1}^{2}(A^{-1})_{jk}\,\mathrm{d}\nu_k = \sum_{k=1}^{2} c_{jk}\,\mathrm{d}\nu_k \tag{12}$$

with A the 2×2 matrix with components $A_{jk} = \int_{a_k} \mathrm{d}\nu_j$. The expansion of the Abelian integral of the vector differential $\mathrm{d}\boldsymbol{\omega}$ at the points at infinity is given by

$$\int_{\infty_1}^{P^{(k)}} \mathrm{d}\boldsymbol{\omega} = \boldsymbol{U}^{(k)} + \boldsymbol{V}^{(k)}\xi + \boldsymbol{W}^{(k)}\xi^2 + \cdots \tag{13}$$

as $P^{(k)} \to \infty_k$ and $\boldsymbol{U}^{(k)} = \int_{\infty_1}^{\infty_k} \mathrm{d}\boldsymbol{\omega}$. Straightforward calculations determine

$$\begin{aligned}
\boldsymbol{V}^{(1)} &= -\mathrm{i}\boldsymbol{c}_2\\
\boldsymbol{W}^{(1)} &= \boldsymbol{0}\\
\boldsymbol{V}^{(2,3)} &= \pm\Delta^{-\frac{1}{2}}\left(\mathrm{i}\boldsymbol{c}_1 + a_0^{(2,3)}\boldsymbol{c}_2\right)\\
\boldsymbol{W}^{(2,3)} &= \frac{\mathrm{i}}{2\Delta}\left(2\mathrm{i}a_1^{(2,3)} + \rho_1\right)\boldsymbol{c}_1 + \left(\frac{a_0^{(2,3)}}{2\Delta}\left(2\mathrm{i}a_1^{(2,3)} + \rho_1\right) \pm \frac{a_1^{(2,3)}}{2\Delta^{\frac{1}{2}}}\right)\boldsymbol{c}_2
\end{aligned}$$

where $\boldsymbol{c}_k = (c_{1k}, c_{2k})^{\mathsf{T}}$.

Using the method discussed in [1], the following genus two finite gap solution to equation (1) may be derived:

$$\begin{aligned}
q_1(x,t) &= \chi_1\frac{\theta(\boldsymbol{g}(x,t) - \boldsymbol{e} + \boldsymbol{r}^{(2)})}{\theta(\boldsymbol{g}(x,t) - \boldsymbol{e})}\exp(E_1x + N_1t)\\
q_2(x,t) &= \chi_2\frac{\theta(\boldsymbol{g}(x,t) - \boldsymbol{e} + \boldsymbol{r}^{(3)})}{\theta(\boldsymbol{g}(x,t) - \boldsymbol{e})}\exp(E_2x + N_2t)
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\chi_k &= \mathrm{i}\delta_k\frac{\theta(\boldsymbol{e})}{\theta(\boldsymbol{r}^{(k+1)} - \boldsymbol{e})},\\
\boldsymbol{r}^{(k)} &= \int_{\infty_1}^{\infty_k}\mathrm{d}\boldsymbol{\omega},\\
\boldsymbol{e} &= \sum_{j=1}^{2}\int_{\infty_1}^{P_j}\mathrm{d}\boldsymbol{\omega} - \boldsymbol{K}.
\end{aligned}$$

Here θ is the Riemann theta function, defined for any $\boldsymbol{y} \in \mathbb{C}^2$ by

$$\theta(\boldsymbol{y}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2}\exp\left(\frac{1}{2}\boldsymbol{m}^{\mathsf{T}}B\boldsymbol{m} + \boldsymbol{m}^{\mathsf{T}}\boldsymbol{y}\right), \tag{15}$$

with B the 2×2 period matrix with components $B_{jk} = \int_{b_k} \mathrm{d}\omega_j$. $\boldsymbol{K}$ is the vector of Riemann constants with base point ∞_1, $\mathcal{D} = P_1 + P_2$ is a divisor of general position and δ_k are real constants—see [1, 8] for further details. $\boldsymbol{g}(x,t) = \boldsymbol{V}x + \boldsymbol{W}t$ with $\boldsymbol{V}, \boldsymbol{W}$ calculated using the Riemann bilinear relations:

$$\begin{aligned}
\boldsymbol{V} &= \frac{\mathrm{i}}{2}(\boldsymbol{V}^{(1)} - \boldsymbol{V}^{(2)} - \boldsymbol{V}^{(3)}) = \boldsymbol{c}_2,\\
\boldsymbol{W} &= \mathrm{i}(\boldsymbol{W}^{(1)} - \boldsymbol{W}^{(2)} - \boldsymbol{W}^{(3)}) = \boldsymbol{0},
\end{aligned} \tag{16}$$

for $k = 1, 2$. Importantly, the constants N_k may also be calculated using these relations and are given by $N_k = \mathrm{i}a_k$, $k = 1, 2$. The constants E_k may also be shown purely imaginary.

The square intensity is given by

$$\boldsymbol{q}^{\dagger}\boldsymbol{q} = \partial_x^2 \ln\theta(\boldsymbol{c}_2 x - \boldsymbol{e}) + R, \tag{17}$$

for some real constant R, via a straightforward extension of the analogous result for the scalar nonlinear Schrödinger equation (SNLS), presented in [8]. The form of the right-hand side of (17) crops up in several quasiperiodic integrable p.d.e's such as the SNLS and Korteweg–de Vries equations.

Since $\boldsymbol{W} = \boldsymbol{0}$, it follows that the solutions for q_1 and q_2, (14), are separable in x and t and therefore necessarily take the form:

$$\begin{aligned} q_1(x,t) &= Q_1(x)\exp\left\{\mathrm{i}a_1 t + \mathrm{i}C_1\int^x \frac{\mathrm{d}x'}{Q_1^2(x')}\right\}, \\ q_2(x,t) &= Q_2(x)\exp\left\{\mathrm{i}a_2 t + \mathrm{i}C_2\int^x \frac{\mathrm{d}x'}{Q_2^2(x')}\right\}, \end{aligned} \tag{18}$$

with $Q_k(x)$ real functions of x and C_k real constants. As shown in [3] solutions of this kind transform (1) to a Hamiltonian system with independent variable x and Hamiltonian:

$$H = \sum_{k=1}^{2}\left(P_k^2 - a_k Q_k^2 + \frac{C_k^2}{Q_k^2}\right) + \left(Q_1^2 + Q_2^2\right)^2 \tag{19}$$

where $P_k = Q_{k,x}$. A second independent integral of the motion is given by

$$\begin{aligned} G = {} & (P_1Q_2 - P_2Q_1)^2 + \left(a_1a_2 - a_2Q_1^2 - a_1Q_2^2\right)\left(Q_1^2 + Q_2^2\right) \\ & - \left(a_2P_1^2 + a_1P_2^2\right) - \left(a_2\frac{C_1^2}{Q_1^2} + a_1\frac{C_2^2}{Q_2^2}\right) + \left(C_1^2\frac{Q_2^2}{Q_1^2} + C_2^2\frac{Q_1^2}{Q_2^2}\right), \end{aligned} \tag{20}$$

and is a consequence of rotational symmetry in the system. Analysis of the above system using spectral techniques produces the results

$$\begin{aligned} Q_1^2 &= \frac{a_1^2 - a_1\wp_{22}(\boldsymbol{u}) - \wp_{12}(\boldsymbol{u})}{a_1 - a_2}, \\ Q_2^2 &= \frac{a_2^2 - a_2\wp_{22}(\boldsymbol{u}) - \wp_{12}(\boldsymbol{u})}{a_2 - a_1}. \end{aligned} \tag{21}$$

Here $\boldsymbol{u} = \left(u_1^{(0)}, x + u_2^{(0)}\right)^{\mathsf{T}}$ and

$$\wp_{jk}(\boldsymbol{u}) = -\partial_{u_j}\partial_{u_k}\ln\sigma(\boldsymbol{u})$$

with the σ-function defined by

$$\sigma(\boldsymbol{u}) = c\exp\{\boldsymbol{u}^{\mathsf{T}}\eta(2\omega)^{-1}\boldsymbol{u}\}\theta[\varepsilon](\boldsymbol{u}). \tag{22}$$

Interestingly, formulae for $|q_1|^2$ and $|q_2|^2$ in the form (21) do not follow directly from the techniques of [1]. In order to compare solutions we sum equations (21) to give

$$\boldsymbol{q}^{\dagger}\boldsymbol{q} = a_1 + a_2 - \wp_{22}(\boldsymbol{u}), \tag{23}$$

which is clearly consistent with the form (17).

It is an elementary corollary of the Riemannn–Roch theorem that every Riemann surface of genus two is hyperelliptic, [10]. It follows that C_A must be birationally equivalent to a curve

$$\left\{(\lambda,\nu)\in\mathbb{CP}^2 : \nu^2 - \sum_{k=0}^{5}\alpha_k\lambda^k = 0\right\} \tag{24}$$

for some $\alpha_k \in \mathbb{C}$ with $\alpha_5 \neq 0$. The explicit form of the birational map is

$$
\begin{aligned}
z &= \frac{\frac{1}{2}\left(\lambda - \frac{1}{3}(a_1+a_2)\right)\rho_1 + \frac{\mathrm{i}}{2}(\nu - \eta_1)}{\left(\lambda - \frac{1}{3}(a_1+a_2)\right)^2 - \rho_2\left(\lambda - \frac{1}{3}(a_1+a_2)\right) + \mathrm{i}\eta_2} \\
w &= \mathrm{i}\left(\lambda - \frac{1}{3}(a_1+a_2)\right).
\end{aligned} \tag{25}
$$

Considering (6) as a quadratic in z, it is clear that $(z, w) \mapsto w$ is a two-sheeted covering of $\mathbb{C}$ by C_A. This is why w depends so simply on λ above. Not only does (25) map the spectral curve C_A into the form (24) of the spectral curve in [3], it precisely recovers the functional dependence of the coefficients α_k on $q(x, t)$ and its derivatives, up to a re-scaling of the constants C_k. These coefficients may be expressed in terms of the conserved quantities H and G as follows:

$$
\begin{aligned}
\alpha_5 &= 4,\\
\alpha_4 &= -8(a_1+a_2),\\
\alpha_3 &= -4H + 4(a_1+a_2)^2 + 8a_1a_2,\\
\alpha_2 &= 4H(a_1+a_2) - 4G - 4C_1^2 - 4C_2^2 - 8a_1a_2(a_1+a_2),\\
\alpha_1 &= 4G(a_1+a_2) - 4a_1a_2H + 8C_1^2a_2 + 8C_2^2a_1 + 4a_1^2a_2^2,\\
\alpha_0 &= -4a_1a_2G - 4C_1^2a_2^2 - 4C_2^2a_1^2.
\end{aligned}
$$

In order to calculate the α_k, the coefficients in the original polynomial (6) have been expressed in terms of a_k, C_k, H and G and are given by

$$
\begin{aligned}
\rho_2 &= (a_1+a_2)/3,\\
\rho_1 &= -2(C_1+C_2),\\
\rho_0 &= (a_1+a_2)^2/9 - (2a_1-a_2)(2a_2-a_1)/9 + H,\\
\eta_2 &= -\mathrm{i}(2a_1-a_2)(2a_2-a_1)/9,\\
\eta_1 &= 2\mathrm{i}C_1(2a_2-a_1)/3 + 2\mathrm{i}C_2(2a_1-a_2)/3,\\
\eta_0 &= -\mathrm{i}(a_1+a_2)(2a_1-a_2)(2a_2-a_1)/27 - 2\mathrm{i}C_1C_2 + \mathrm{i}G + 2\mathrm{i}(a_1+a_2)H/3.
\end{aligned}
$$

The birational equivalence of the spectral curves is important because it means that the fields of meromorphic functions on the two curves are isomorphic.

Applying the relations $\nu = f_z(z, w)$ and $\mathrm{d}f = f_z\,\mathrm{d}z + f_w\,\mathrm{d}w = 0$, the differentials (11) are given in terms of λ and ν by

$$
\mathrm{d}\nu_1 = \mathrm{d}u_1, \qquad \mathrm{d}\nu_2 = \mathrm{d}u_2 - \tfrac{1}{3}(a_1+a_2)\,\mathrm{d}u_1, \tag{26}
$$

where $\mathrm{d}u_1 = \mathrm{d}\lambda/\nu$ and $\mathrm{d}u_2 = \lambda\,\mathrm{d}\lambda/\nu$ are the differentials used in [3]. Given that our choice of first homology basis on the three-sheeted surface has been free up until now, we choose it such that a and b cycles are directly transferred between two and three-sheeted coverings. This means we have

$$
(2\omega)^{-1} = \frac{1}{2\pi\mathrm{i}}\begin{pmatrix} c_{11} - \frac{1}{3}(a_1+a_2) & c_{12} \\ c_{21} - \frac{1}{3}(a_1+a_2) & c_{22} \end{pmatrix} \tag{27}
$$

where $(2\omega)_{jk} = \int_{a_k} \mathrm{d}u_j$ using the notation of [3]. Thus it is clear that (17) and (23) are the same up to a choice of divisor.

It is natural to ask at this stage whether any of the higher genus spectral curves of the VNLS solutions are hyperelliptic. Each of the spectral polynomials in [1] is of the form (6) but with the degrees of $P(z)$ and $Q(z)$ dependent on the genus. Thus, thinking of (6) as a cubic polynomial in w, one sees that the function $(z, w) \mapsto z$ is a function of degree three on the corresponding Riemann surface. A hyperelliptic Riemann surface of genus g does not support functions whose degree is odd and less than or equal to g, [10]. It is therefore clear that spectral curves of the VNLS with genus greater than two are not hyperelliptic.

We have shown how, by allowing the constants C_{kk} to be nonzero, the method of [1] may be employed in calculating a new explicit formula for genus two solutions to the VNLS. Since these solutions are separable one ought to be able to obtain them using the method described in [3]. Indeed we have demonstrated that the two methods do yield the same solutions, and have given explicitly a birational mapping between the spectral curve arising via the 3×3 Lax system (2) and that of the 2×2 Lax representation in [3]. We have thus tied together two fundamentally different approaches to finding explicit formulae for finite gap solutions to the VNLS. In future publications we intend to generalize the method of [1] to encompass solutions of arbitrary genus by employing the idea described above—introducing constants appended to the nth degree polynomial $\mathcal{L}_n(z)$ in the Lax representation (2). In this way we expect to be able to produce formulae of the form (14) for arbitrary even genus.

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